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Operator means and comparison of their norms: general theory and examples

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The classical Heinz inequality ([4]) states

$$||H^{\theta}XK^{1-\theta}+H^{1-\theta}XK^{\theta}||\leq||HX+XK|| \quad (0\leq\theta\leq 1) \tag{1}$$

for Hilbert space operators H, K, X with $H, K\geq 0$. This inequality remains valid for an arbitrary unitarily invariant norm $|||\cdot|||$, and the special case $\theta=1/2$ of this generalized version is the "arithmetic-geometric mean" inequality obtained in [1]:

$$|||H^{1/2}XK^{1/2}|||\leq\frac{1}{2}|||HX+XK|||.$$

In recent years such operator (and/or matrix) means and comparison of their norms are under active investigation (see [2, 5, 6, 8, 10] for instance). We will briefly explain the general apparatus (obtained in [7]) to deal with such problems. More details as well as a more complete list of references can be found in my survey article in "Sugaku" to be published shortly (or in [7]).

1. Operator (matrix) means

In this article a scalar (symmetric homogeneous) mean will mean a continuous function $M(s,t)$ on on $[0,\infty)\times[0,\infty)$ satisfying

(a) $M(s,t)=M(t,s)$,
(b) $M(\alpha s,\alpha t)=\alpha M(s,t)$ for $\alpha\geq 0$,
(c) $M(s,t)$ increasing in each variable,
(d) $\min\{s,t\}\leq M(s,t)\leq\max\{s,t\}$.

The set of all such means will be denoted by $\mathfrak{M}$. Typical examples are

$$(st)^{1/2},\quad (s-t)/(\log s-\log t)\ \left(=\int_0^1 s^xt^{1-x}dx\right),$$
$$\frac{s+t}{2},\quad \frac{s^{\theta}t^{1-\theta}+s^{1-\theta}t^{\theta}}{2}\ (\text{with } 0\leq\theta\leq 1).$$

To each $M(s,t)\in\mathfrak{M}$ a corresponding operator mean (denoted by $M(H,K)X$) will be associated.

To get more intuition on the subject matter, we begin with the matrix case ($H,K,X\in M_n(\mathbf{C})$ and $H,K\geq 0$). At first we diagonalize H, K:

$$H=U\mathrm{diag}(t_1,t_2,\cdots,t_n)U^*,\quad K=V\mathrm{diag}(s_1,s_2,\cdots,s_n)V^*$$

with unitary matrices U, V. For each $M\in\mathfrak{M}$ we define

$$M(H,K)X=U\Big(\big[M(s_i,t_j)\big]_{i,j=1,2,\cdots,n}\circ(U^*XV)\Big)V^*$$

with the Schur product $\circ$. If $M(s,t)$ is of the form $\sum_{k=1}^{\ell} f_k(s)g_k(t)$, we simply have

$$M(H,K)X = \sum_{k=1}^{\ell} f_k(H)Xg_k(K).$$

Let us consider the projections $P_i = UE_{ii}U^*, Q_j = VE_{jj}V^*$. Then, $H = \sum_{i=1}^{n} s_i P_i$, $K = \sum_{j=1}^{n} t_j Q_j$ are the spectral decompositions of H, K, and we observe that the above matrix mean $M(H,K)X$ can be also expressed as

$$M(H,K)X = \sum_{i,j=1}^{n} M(s_i,t_j)P_iXQ_j.$$

We now move to the general (operator) case. Let H, K be positive operators with the spectral decompositions

$$H = \int_0^{\|H\|} s\, dE_s, \quad K = \int_0^{\|K\|} t\, dF_t.$$

The above expression involving $\sum_{ij}$ suggests that an operator mean $M(H,K)X$ should be something like

$$M(H,K)(X) = \int_0^{\|H\|}\int_0^{\|K\|} M(s,t)\, dE_s X dF_t$$

(at least formally). Of course the meaning of this double integral has to be justified, however fortunately the well-developed theory of Stieltjes double integral transformations (see the recent survey article [3]) is at our disposal. Also the problem on multipliers has to be taken care of.

2. Stieltjes double integral transformations

Given a function $\phi(s,t) \in L^\infty([0,\|H\|]\times[0,\|K\|];\lambda\times\mu)$ (with $\lambda \sim dE_s$ and $\mu \sim dF_t$ in the absolute continuity sense), we would like to make a sense out of

$$\Phi(X) = \int_0^{\|H\|}\int_0^{\|K\|} \phi(s,t)\, dE_s X dF_t \ (\in B(\mathcal{H})) \text{ for } X \in B(\mathcal{H}). \tag{2}$$

We begin with the case $X \in \mathcal{C}_2(\mathcal{H})$, the Hilbert-Schmidt operators. For Borel subsets

$$\Lambda \subseteq [0,||H||], \quad \Xi \subseteq [0,||K||]$$

$\pi_\ell(E_\Lambda), \pi_r(F_\Xi)$ are commuting projections acting on $\mathcal{C}_2(\mathcal{H})$, where $\pi_\ell(\cdot)$, $\pi_r(\cdot)$ mean the left and right multiplications acting on $\mathcal{C}_2(\mathcal{H})$. Let us consider the correspondence

$$\Lambda \times \Xi \subseteq [0,||H||] \times [0,||K||] \longrightarrow \pi_\ell(E_\Lambda)\pi_r(F_\Xi) \in B(\mathcal{C}_2(\mathcal{H}))_{proj},$$

which can be easily extended to a spectral family. Therefore, we can perform functional calculus relative to this spectral family. Thus, we have

$$\Phi(\cdot) = \int_0^{\|H\|}\int_0^{\|K\|} \phi(s,t)\, d(\pi_\ell(E_\cdot)\pi_r(F_\cdot)) \in B(\mathcal{C}_2(\mathcal{H})),$$

and $\Phi(X)$ $(\in \mathcal{C}_2(\mathcal{H}))$ makes a perfect sense. The right side of (2) should be understood in this way (for $X \in \mathcal{C}_2(\mathcal{H})$).

The discussion so far is indeed the starting point of the theory of Stieltjes double transformations by Birmann-Solomjak. For each practical purpose the definition domain of $\Phi(\cdot)$ should be enlarged as much as possible (to $\mathcal{C}_p(\mathcal{H}), \mathcal{C}_1(\mathcal{H}), B(\mathcal{H})$, etc. depending upon available regularity assumption). Various important applications to many subjects (such as perturbation theory, Volterra operators, Hankel operators and so on) are known.

Definition. $\phi(s,t)$ is called a Schur multiplier (more precisely, $\mathcal{C}_1$-Schur multiplier relative to (H,K)) when $\Phi(\mathcal{C}_1(\mathcal{H})) \subseteq \mathcal{C}_1(\mathcal{H})$.

Theorem (V.V. Peller's characterization, [9])

For $\phi \in L^\infty([0,\|H\|] \times [0,\|K\|]; \lambda \times \mu)$ the following conditions are all equivalent:

(i) ϕ is a Schur multiplier (relative to (H,K));

(ii) whenever a measurable function $k : [0,\|H\|] \times [0,\|K\|] \to \mathbf{C}$ is the kernel of a trace class operator $L^2([0,\|H\|];\lambda) \to L^2([0,\|K\|];\mu)$, so is the product $\phi(s,t)k(s,t)$;

(iii) one can find a finite measure space (Ω,σ) and functions $\alpha \in L^\infty([0,\|H\|] \times \Omega; \lambda \times \sigma)$, $\beta \in L^\infty([0,\|K\|] \times \Omega;\ \mu \times \sigma)$ satisfying

$$\phi(s,t) = \int_\Omega \alpha(s,x)\beta(t,x)d\sigma(x); \tag{3}$$

(iv) one can find a measure space (Ω,σ) and measurable functions α,β on $[0,\|H\|] \times \Omega$, $[0,\|K\|] \times \Omega$ respectively satisfying (3) and

$$\left\| \int_\Omega |\alpha(\cdot,x)|^2 d\sigma(x) \right\|_{L^\infty(\lambda)} \left\| \int_\Omega |\beta(\cdot,x)|^2 d\sigma(x) \right\|_{L^\infty(\mu)} < \infty.$$

When $\phi(s,t)$ is a Schur multiplier, $\Phi : \mathcal{C}_1(\mathcal{H}) \to \mathcal{C}_1(\mathcal{H})$ is a bounded linear operator (by the closed graph theorem) so that we have the transpose ${}^t\Phi : B(\mathcal{H}) = \mathcal{C}_1(\mathcal{H})^* \to B(\mathcal{H}) = \mathcal{C}_1(\mathcal{H})^*$. Starting from the decomposition (3), one can prove

$$\int_0^{\|H\|}\int_0^{\|K\|} \phi(s,t)\, dE_s X dF_t = \int_\Omega \alpha(H,x)X\beta(K,x)d\sigma(x).$$

3. Norm inequalities for operator means

When a scalar mean $M(s,t)$ ($\in \mathfrak{M}$) is a Schur multiplier, we define

$$M(H,K)X = \int_0^{\|H\|}\int_0^{\|K\|} M(s,t)\, dE_s X dF_t \in B(\mathcal{H}) \quad (\text{for each } X \in B(\mathcal{H})).$$

Theorem (F. Hiai and H. Kosaki, [6, 7])

For $M, N \in \mathfrak{M}$ the following conditions are all equivalent:

(i) There exists a symmetric probability measure ν on $\mathbf{R}$ with the following property: if N is a Schur multiplier relative to (H,K) of non-singular positive operators, then so is M and

$$M(H,K)X = \int_{-\infty}^{\infty} H^{ix}(N(H,K)X)K^{-ix}d\nu(x) \text{ for } X \in B(\mathcal{H});$$

(ii) If N is a Schur multiplier relative to a pair (H, K) of positive operators, then so is M and

$$|||M(H,K)X||| \leq |||N(H,K)X|||$$

for all unitarily invariant norms and all $X \in B(\mathcal{H})$;

(iii) $\|M(H,H)X\| \leq \|N(H,H)X\|$ for all X of finite rank and for all $H \geq 0$;

(iv) For each n and $\lambda_1, \lambda_2, \cdots, \lambda_n > 0$

$$\left[\frac{M(\lambda_i, \lambda_j)}{N(\lambda_i, \lambda_j)}\right]_{i,j=1,2,\cdots,n} \geq 0;$$

(v) $M \prec N$, i.e., $\dfrac{M(e^x, 1)}{N(e^x, 1)}$ is a positive definite function.

This theorem explains the Heinz inequality (1) as follows: We set

$$M(s,t) = \frac{s^{\frac{1+\alpha}{2}} t^{\frac{1-\alpha}{2}} + s^{\frac{1-\alpha}{2}} t^{\frac{1+\alpha}{2}}}{2}, \quad N(s,t) = \frac{s+t}{2} \quad (\alpha \in [0,1]).$$

The ratio is

$$\frac{M(e^x, 1)}{N(e^x, 1)} = \frac{e^{\left(\frac{1+\alpha}{2}\right)x} + e^{\left(\frac{1-\alpha}{2}\right)x}}{e^x + 1} = \frac{\cosh(\alpha x/2)}{\cosh(x/2)}$$

whose Fourier transform is given by

$$\int_{-\infty}^{\infty} \frac{\cosh(\alpha x/2)}{\cosh(x/2)} e^{ixy} dx = \frac{4\pi \cosh(\pi y) \cos(\alpha\pi/2)}{\cosh(2\pi y) + \cos(\alpha\pi)} > 0.$$

Bochner's theorem thus yields $M \prec N$.

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